Percolation and the Potts Model

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The Kasteleyn–Fortuin formulation of bond percolation as a lattice statistical model is rederived using an alternate approach. It is shown that the quantities of interest arising in the percolation problem, including the critical exponents, can be obtained from the solution of the Potts model. We also establish the Griffith inequality for critical exponents for the bond percolation problem.

KEY WORDS: Percolation; Potts model; Griffiths inequality.

1. INTRODUCTION

The percolation process provides a simple picture of a critical point transition and has been of increasing recent theoretical interest. We refer to several review articles[1–3] for a general survey of the subject. An important development first established by Kasteleyn and Fortuin [4,5] is the connection between bond percolation and a lattice statistical model. This consideration leads to a formulation of the percolation problem which is extremely useful, for many of the techniques readily available in statistical mechanics can now be applied to percolation (see, e.g., Ref. 6). However, much of this otherwise elegant result appears to be masked under the formality of the graph-theoretic approach of Kasteleyn and Fortuin, and we feel it worthwhile to have an alternate derivation to elucidate the situation. We present an approach to the Kasteleyn–Fortuin formulation of bond percolation which we believe to be simpler and more direct; it also permits a straightforward extension of the Griffiths inequality to the percolation problem. In Section 2 we review the bond percolation problem for the purpose of establishing the notation. The Potts model is introduced in Section 3, and we show that the quantities of interest arising in the percolation problem, including the critical exponents,

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can be obtained from the solution of the Potts model. Expressions showing these relationships are explicitly given. The Griffiths exponent inequality for the percolation process is established in Section 4.

2. BOND PERCOLATION

Consider a lattice (or graph) G composed of N sites (vertices) and M edges (lines). The graph does not have to be regular, although consideration in practice is always confined to regular lattices and for M and N large, with the ratio

$$z = \lim_{M,N \to \infty} M/N \tag{1}$$

finite. An example is $z = 2$ for the square lattice.

In a bond percolation process, there is a probability p for each edge of G to be "occupied" independently and a probability $1 - p$ for it to be "vacant." It is convenient to picture the occupied edges as covered by bonds placed along the edges. Two sites that are connected by a chain of occupied edges, or bonds, are said to belong to the same cluster. Then, one of the first questions that can be raised about this percolation process is the probability that a randomly chosen site, say, the origin, of an infinite lattice belongs to a cluster of infinite size. This is the percolation probability $P(p)$.

It is clear that $P(0) = 0$ and $P(1) = 1$. The interesting property of $P(p)$ is that it remains zero for p less than a certain critical value p_c, and rises sharply at p_c with the behavior[3]

$$P(p) \sim (p - p_c)^\beta, \qquad p \simeq p_c+ \tag{2}$$

for some positive β. This defines the critical exponent β for the percolation process.

To formulate the problem mathematically, observe that each bond configuration of G is conveniently represented by a subgraph G' of G whose edge set contains precisely the occupied edges. Let $A \equiv A(G')$ be a number associated with some property of the subgraph G'. Examples are

$$e \equiv e(G') = \text{number of occupied edges in } G'$$
$$n \equiv n(G') = \text{number of clusters in } G' \tag{3}$$

The probability for the configuration G' to occur is

$$\pi(G') = p^e(1 - p)^{M-e} \tag{4}$$

The average of the quantity A is then defined to be

$$\langle A \rangle = \sum_{G'} \pi(G')A(G') \tag{5}$$

As in statistical mechanics, the average quantities of interest are usually extensive, and we shall need the "thermodynamic" limit

$$\langle A \rangle_0 = \lim_{N \to \infty} N^{-1} \langle A \rangle \tag{6}$$

Generally let $A_c \equiv A_c(G')$ denote some property of a cluster c in G', and $A_0 \equiv A_0(G')$ the property of the cluster containing the origin. Examples are s_c and s_0, the number of sites in a cluster, and b_c and b_0, the number of bonds in a cluster. Then, by translational symmetry, we have the identity[7]

$$\langle A_0 \rangle = \left\langle \sum_c s_c A_c \right\rangle_0 \tag{7}$$

The factor s_c in (7) arises because there are precisely s_c ways for the origin to lie within a cluster of s_c sites. If we define

$$\gamma_0 = 1, \quad \text{if } s_0 \text{ is finite}$$
$$= 0, \quad \text{otherwise}$$

It then follows from our definitions that

$$P(p) \equiv 1 - \langle \gamma_0 \rangle = 1 - \left\langle \sum_c{}' s_c \right\rangle_0 \tag{8}$$

where the prime restricts the summation to finite clusters.

Other quantities of interest include $S(p)$, the mean size of the *finite* cluster that contains the origin. In our notation,

$$S(p) \equiv \langle s_0 \gamma_0 \rangle = \left\langle \sum_c{}' s_c^2 \right\rangle_0 \tag{9}$$

Here, we have adopted the convention of specifying the cluster size by its site content. Alternately, as is customary in bond percolation considerations, we may also specify the cluster size by its bond content. The mean cluster size is then taken to be

$$S^{(B)}(p) = (zp)^{-1} \left\langle \sum_c{}' b_c^2 \right\rangle_0 \tag{10}$$

It can be shown[8] that (9) and (10) give rise to the same exponents γ and γ':

$$S(p) \sim (p - p_c)^{-\gamma}, \quad p \simeq p_c -$$
$$\sim (p_c - p)^{-\gamma'}, \quad p \simeq p_c + \tag{11}$$

Here the primed exponent refers to $p \simeq p_c +$, which is analogous to $T \simeq T_c -$ of the ordinary critical point.

Another quantity of interest is the mean number of clusters per site

$$G(p) = \langle n \rangle_0 \tag{12}$$

Since infinite clusters, whenever they appear, are limited in numbers, there is presumably no need to distinguish $G(p)$ and $G^{(F)}(p)$, the mean number of *finite* clusters. This is certainly so for $p \leqslant p_c$. In evaluating (12), the isolated sites are considered as individual clusters. The behavior of the singular part of $G(p)$ now defines the critical exponents α and α':

$$G_{\text{sing}}(p) \sim |p - p_c|^{2-\alpha, 2-\alpha'}, \qquad p \simeq p_c \tag{13}$$

Finally, the pair connectivity $c(\mathbf{r}, p)$ is defined to be the probability that the sites at the origin and at $\mathbf{r}$ are connected (i.e., they belong to the same cluster). This connectivity can be written as

$$c(\mathbf{r}, p) = \langle \gamma(\mathbf{r}) \rangle \tag{14}$$

where

$$\gamma(\mathbf{r}) = 1, \qquad \text{if the sites at the origin and } \mathbf{r} \text{ are connected}$$
$$= 0, \qquad \text{otherwise}$$

The decay of the connectivity at p_c defines the exponent η for the percolation process:

$$c(\mathbf{r}, p_c) \sim r^{-(d-2+\eta)}, \qquad r \to \infty \tag{15}$$

Here d is the dimensionality of the lattice.

It is also customary to consider $c^{(F)}(\mathbf{r}, p)$, the probability that the sites at the origin and at $\mathbf{r}$ belong to the same *finite* cluster.[7] We expect $c(\mathbf{r}, p)$ and $c^{(F)}(\mathbf{r}, p)$ to be identical for $p \leqslant p_c$.

The Fourier transform of the pair connectivity takes the form

$$\tilde{c}(k, p) \equiv \sum_{\mathbf{r}} c(\mathbf{r}, p) \exp(-i\mathbf{k} \cdot \mathbf{r}) = \left\langle \sum_{\mathbf{r} \in c_0} \exp(-i\mathbf{k} \cdot \mathbf{r}) \right\rangle \tag{16}$$

where the summation in (16) extends over all sites in the cluster c_0 containing the origin. By expanding the exponential and averaging over all directions of $\mathbf{k}$, one obtains the moment expansion[9]

$$\tilde{c}_{\text{av}}(k, p) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n+1)!} \mu_{2n}(p) \tag{17}$$

where

$$\mu_n(p) = \left\langle \sum_{\mathbf{r} \in c_0} r^n \right\rangle \tag{18}$$

There is strong evidence that $\mu_n(p)$ diverges as[9]

$$\mu_n(p) \sim |p_c - p|^{-\gamma - n\nu}, \qquad p \approx p_c - \tag{19}$$

which defines the critical exponents ν. The exponent ν' can be defined similarly from the critical behavior of the corresponding $\mu_n^{(F)}(p)$ at $p_c +$. The

definitions of the exponents δ, Δ, and Δ' for the percolation process will be given in the next section.

3. THE POTTS MODEL

The Potts model[10] is a generalization of the Ising model so that the spins can be in one of q states ($q = 2$ is the Ising model). Let the spin state at the ith site be specified by $\xi_i = 1, 2, \ldots, q$. The Hamiltonian for the Potts model in an external field $-H$ is

$$\mathcal{H} = -\epsilon \sum_{\langle ij \rangle} \delta_{kr}(\xi_i, \xi_i) - H \sum_i \delta_{kr}(\xi_i, \alpha) \tag{20}$$

where the first summation extends over the M edges of G, and the external field $-H$ is applied to the spin state α. The partition function of the Potts model now reads

$$Z(q; k, L) = \sum_{\xi_i = 1}^{q} \prod_{\langle ij \rangle} [1 + v\delta_{kr}(\xi_i, \xi_j)] \prod_i [1 + u\delta_{kr}(\xi_i, \alpha)] \tag{21}$$

where

$$v = e^K - 1, \qquad u = e^L - 1, \qquad K = \epsilon/kT, \qquad L = H/kT \tag{22}$$

Following Baxter[11] and Ref. 5, we expand the first product in (21) and use the subgraphs of G to represent the terms in the expansion. Each term in the expansion is conveniently represented by a subgraph G' whose edge set coincides with the $v\delta_{kr}(\xi_i, \xi_j)$ factors contained in the term. For a given G' of the expansion, we further expand the second product in (21) for *each cluster*. For the first term, viz. 1, in the expansion for a cluster, the summations in (21) yield a factor q. For the remaining $2^{s_c} - 1$ terms of the cluster, which contains s_c sites, the summations yield a factor $(1 + u)^{s_c} - 1 = e^{Ls_c} - 1$. It follows that (21) takes the form

$$Z(q; K, L) = \sum_{G'} v^e \prod_c (e^{Ls_c} + q - 1) \tag{23}$$

Comparing (23) with (5), we see that we can write

$$Z(q; K, L) = e^{MK} \left\langle \prod_c (e^{Ls_c} + q - 1) \right\rangle \tag{24}$$

provided that, of course, we take $v = p/(1 - p)$, or

$$p = 1 - e^{-K} \tag{25}$$

Equation (24) is the basic relation connecting the Potts model with the bond percolation problem. Note that while the Potts partition function (21)

is defined strictly for positive, integral q, the expression (23) or (24) provides a natural continuation of the partition function to other values of q. This leads to the random cluster model of Refs. 4 and 5. For our purposes, it suffices to start from the solution of the Potts model and simply treat the parameter q occurring in the solution as a continuous variable. This permits us to carry out operations such as derivatives with respect to q. Now write the free energy per site of the Potts model as

$$f(q; K, L) = \lim_{N \to \infty} N^{-1} \ln Z(q; K, L) \tag{26}$$

and further define

$$h(K, L) = \left[\frac{\partial}{\partial q} f(q; K, L) \right]_{q=1} \tag{27}$$

It is easy to verify that, after an interchange of the order of the derivative and the thermodynamic limit,

$$h(K, L) = \left\langle \sum_c e^{-Ls_c} \right\rangle_0 = \left\langle \sum_c{}' e^{-Ls_c} \right\rangle_0, \qquad L > 0 \tag{28}$$

While for $L = 0$ the summation in (28) ranges over all clusters, the summation is, in effect, restricted to clusters of finite size for any $L > 0$. We then obtain from (12) and (28) the identities

$$G(p) = h(K, 0), \qquad G^{(F)}(p) = h(K, 0+) \tag{29}$$

Therefore, $G(p) = G^{(F)}(p)$ if and only if $h(K, L)$ is continuous at $L = 0$. Further define

$$P(p, L) \equiv 1 + (\partial/\partial L)h(K, L) \tag{30}$$

Comparison of (8) and (28) then establishes the identity

$$P(p) = P(p, 0+) \tag{31}$$

Similarly, (9) leads to the expression

$$S(p) = \left[\frac{\partial^2}{\partial L^2} h(K, L) \right]_{L=0+} \tag{32}$$

It is now seen that $h(K, L)$ plays the role of the free energy of a statistical model and we are led to the correspondences[4,5]

$$G(p) \leftrightarrow \text{free energy}$$

$$P(p) \leftrightarrow \text{magnetization}$$

$$S(p) \leftrightarrow \text{susceptibility}$$

Pursuing the analogy further, it is now possible to define the exponent δ for the percolation process from the relation [12]

$$P(p_c, L) \sim L^{1/\delta}, \qquad L \simeq 0 \tag{33}$$

Similarly, we can define the gap exponents Δ and Δ' using

$$\left[\frac{\partial^3}{\partial L^3} h(K, L)\right]_{L=0+} \sim |p - p_c|^{-\gamma-\Delta, -\gamma-\Delta'} \tag{34}$$

The above analysis can be extended if an external field of the form

$$-H_1 \sum_{\langle ij \rangle} \delta_{kr}(\xi_i, \alpha)\, \delta_{kr}(\xi_j, \alpha)$$

is included in the Potts Hamiltonian (20). This changes (24) into

$$Z(q; K, L, L_1) = e^{MK} \left\langle \prod_c (e^{Ls_c + L_1 b_c} + q - 1) \right\rangle \tag{35}$$

where $L_1 = H_1/kT$. Quantities involving the averages of b_c, such as $S^{(B)}(p)$ in (10), can be conveniently expressed as the derivatives of the free energy per site defined by (35). In fact, the analogy between the Potts model and the bond percolation may be developed by considering L_1 instead of L as the external field. Since this does not lead to any new values for the critical exponents,[8] we shall not consider it any further.

Finally, we consider the pair connectivity $c(\mathbf{r}, p)$. Let $P_{\alpha\alpha}(\mathbf{r})$ be the probability in the zero-field Potts model that the sites at the origin and $\mathbf{r}$ are both in the spin state α. Since $P_{\alpha\alpha}(\mathbf{r}) = q^{-2}$ when there is no correlation, the correlation function can be defined to be [13]

$$\Gamma_{\alpha\alpha}(\mathbf{r}) = P_{\alpha\alpha}(\mathbf{r}) - q^{-2} \tag{36}$$

Now

$$P_{\alpha\alpha}(\mathbf{r}) = {\sum}' \prod_{\langle ij \rangle} [1 + v\delta_{kr}(\xi_i, \xi_j)]/Z(q; K, 0) \tag{37}$$

where the prime over the summation denotes that the spins at the origin and at $\mathbf{r}$ are held at state α. Again, we use a graphical representation of terms in the expansion of the numerator in (37). However, it is now necessary to distinguish whether the sites at the origin and at $\mathbf{r}$ are connected. We thus obtain

$$P_{\alpha\alpha}(\mathbf{r}) = q^{-1} \frac{\langle \gamma(\mathbf{r})q^n \rangle}{\langle q^n \rangle} + q^{-2} \frac{\langle [1 - \gamma(\mathbf{r})]q^n \rangle}{\langle q^n \rangle} \tag{38}$$

It follows then that

$$q^2(q - 1)^{-1}\Gamma_{\alpha\alpha}(F) = \langle \gamma(\mathbf{r})q^n \rangle / \langle q^n \rangle \tag{39}$$

or, using (14),

$$c(\mathbf{r}, p) = \left[\frac{\partial}{\partial q} \Gamma_{\alpha\alpha}(\mathbf{r}) \right]_{q=1} \tag{40}$$

Similarly, we expect

$$c^{(F)}(\mathbf{r}, p) = \left[\frac{\partial}{\partial q} \Gamma_{\alpha\alpha}(\mathbf{r}, 0+) \right]_{q=1} \tag{41}$$

where $\Gamma_{\alpha\alpha}(\mathbf{r}, L)$ is an extension of (36) using the Hamiltonian (20). However, the proof of (41) requires an identity for the Potts model similar to that of the equivalence of the large-distance, two-spin correlation and the square of the spontaneous magnetization of the Ising model. This identity has not been rigorously established.

The relations (40) suggests the further analogy [4,5]

$$c(\mathbf{r}, p) \leftrightarrow \text{correlation function}$$

The exponents η, ν, and ν' for the percolation may then be extracted from the Potts correlation function.

4. GRIFFITHS EXPONENT INEQUALITY

Kasteleyn and Fortuin [4,5] have established the Rushbrooke inequality [14]

$$\alpha' + 2\beta + \gamma' \geqslant 2 \tag{42}$$

for the bond percolation. We now show that our formulation of the percolation permits a straightforward derivation of the Griffiths inequality [15]

$$\alpha' + \beta(1 + \delta) \geqslant 2 \tag{43}$$

for the critical exponents for the bond percolation.

We observe from (28) that $g(K, L) \equiv L + h(K, L)$ is convex in L. This implies that the Legendre transformation

$$A(p, P) \equiv g[K, L(K, P)] - L(K, P)P \tag{44}$$

of $g(K, L)$ is concave in P, where $p = 1 - e^{-K}$ and $L(K, P)$ is to be derived from (30). Further, from the existence of $P(p) = P(p, L = 0+)$ and the relation $L = \partial A/\partial P$ we know that $A(p, P)$ is a constant for $p > p_c$ and $P \leqslant P(p)$. These are the basic ingredients needed in the derivation of the Griffiths inequality for a thermodynamic system. Following the standard argument, [16] we are thus led to the Griffiths inequality (43) for the percolation problem. The appearance of α' in the equality is spurious wherever $\alpha' < 0$, which is the case for some percolation problems in two dimensions. [17] For such systems, the Griffiths equality should read

$$\beta(1 + \delta) \geqslant 2 \qquad (\alpha' \leqslant 0) \tag{45}$$

NOTE ADDED IN PROOF

It has been recently established that the site percolation can be formulated as a Potts model with many-body interactions (H. Kunz and F. Y. Wu, to be published in *J. Phys. C*).

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